

Chapter 25. Fusion and Tupling

Zhenjiang Hu, Wei Zhang

School of Computer Science
Peking University

December 14, 2022

Outline

- 1 Fusion
- 2 Tupling

max

Consider the function to compute the maximum of a list (by reusing *sort*):

$$\begin{aligned} \mathit{max} &: [Int] \rightarrow Int \\ \mathit{max} &= \mathit{head} \cdot \mathit{sort} \end{aligned}$$

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where *sort* is defined by

$$\begin{aligned} \mathit{sort} &= \mathit{foldr} \ \mathit{insert} \ [] \\ \mathit{insert} \ a \ [] &= [a] \\ \mathit{insert} \ a \ (b : x) &= \mathbf{if} \ a \geq b \ \mathbf{then} \ a : (b : x) \\ &\quad \mathbf{else} \ b : \mathit{insert} \ a \ x. \end{aligned}$$

How to eliminate all intermediate results in computing *max*?

reverse

Consider the following function to reverse a list:

$$\begin{aligned} \text{rev } x &= \text{fastrev}' x [] \\ \text{fastrev}' x y &= \text{reverse } x ++ y \end{aligned}$$

where

$$\text{reverse} = \text{foldr } (\lambda a r. r ++ [a]) []$$

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Exercise

Show evaluation steps of $\text{rev } [1, 2, 3, 4]$, and explain that $(\text{rev } xs)$ is a quadratic program.

Fusion Law for Foldr

Lemma (Foldr Fusion)

$$\frac{f(a \oplus r) = a \otimes f r}{f \cdot \text{foldr} (\oplus) e = \text{foldr} (\otimes) (f e)}$$

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Or written as

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Fusion: max

Consider the fusion for *max*:

$$\mathit{max} = \mathit{head} \cdot \mathit{foldr} \mathit{insert} []$$

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To apply the foldr fusion lemma, we consider calculation of *head* (*insert a r*).

We calculate as follows.

- For the case of $r = []$, we have:

$$\begin{aligned} & \text{head (insert a [])} \\ = & \quad \{ \text{def. of insert} \} \\ & \text{head [a]} \\ = & \quad \{ \text{def. of head} \} \\ & a \end{aligned}$$

Fusion Example: max

- For the case of $r = b : x$, we have:

$$\begin{aligned}
 & \text{head } (\text{insert } a \ (b : x)) \\
 = & \quad \{ \text{def. of } \text{insert} \} \\
 & \text{head } (\text{if } a \geq b \ \text{then } a : (b : x) \ \text{else } b : \text{insert } a \ x) \\
 = & \quad \{ \text{distribute } \text{head} \ \text{over } \text{if} \} \\
 & \text{if } a \geq b \ \text{then } \text{head } (a : (b : x)) \ \text{else } \text{head } (b : \text{insert } a \ x) \\
 = & \quad \{ \text{def. of } \text{head} \} \\
 & \text{if } a \geq b \ \text{then } a \ \text{else } b \\
 = & \quad \{ \text{assumption: } r = b : x \} \\
 & \text{if } a \geq \text{head } r \ \text{then } a \ \text{else } \text{head } r
 \end{aligned}$$

In summary, we have

$$\begin{aligned} \text{head} (\text{insert } a \ r) &= a \otimes \text{head } r \\ \text{where } a \otimes r &= \text{if } a \geq r \text{ then } a \text{ else } r \end{aligned}$$

It follows from the foldr fusion lemma that we get the following new definition for *max*.

$$\text{max} = \text{foldr } (\otimes) \ (-\infty)$$

A linear time program!

Fusion Example: Fast Reverse

Consider fusion of the following program:

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Fusion Example: Fast Reverse

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$$\mathit{fastrev}'\ x\ y = \mathit{reverse}\ x\ ++\ y$$

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What is the intermediate list in the above computation?

We can see where fusion calculation is application if we rewrite the definition.

$$\begin{aligned} \mathit{fastrev}'\ x &= (\mathit{++})\ (\mathit{reverse}\ x) \\ &= \underline{((\mathit{++}) \cdot \mathit{foldr}\ (\lambda a\ r.\ r\ ++\ [a])\ [])}\ x \end{aligned}$$

Let us calculate the fusion condition:

$$\begin{aligned}
 & (\#) (r \# [a]) \\
 = & \quad \{ \eta \text{ expansion} \} \\
 & \lambda y. (\#) (r \# [a]) y \\
 = & \quad \{ \text{section notation} \} \\
 & \lambda y. (r \# [a]) \# y \\
 = & \quad \{ \text{associativity of } \# \} \\
 & \lambda y. r \# ([a] \# y)
 \end{aligned}$$

Marching it with $a \otimes ((\#) r)$ gives

$$a \otimes r' = \lambda y. r' ([a] \# y)$$

So we get

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That is,

$$\begin{aligned} \mathit{fastrev}'\ []\ y &= y \\ \mathit{fastrev}'\ (a : r)\ y &= \mathit{fastrev}'\ r\ (a : y) \end{aligned}$$

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A linear time algorithm!

Homework BMF 3-1

Using the foldr fusion lemma to prove the following two equations.

- 1 $foldr (\oplus) e \cdot map f = foldr (\lambda a r. f a \oplus r) e$
- 2 $map f \cdot map g = map (f \cdot g)$

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Exercise

What is the time complexity for the following function that computes the maximum element from a list.

$$\begin{aligned} \mathit{maximum} [a] &= a \\ \mathit{maximum} (a : x) \mid a > \mathit{maximum} x &= a \\ &\mid \mathit{otherwise} &= \mathit{maximum} x \end{aligned}$$

Enumerating Bigger Elements

Enumerate all bigger elements in a list. An element is bigger if it is greater than the sum of the elements that follow it till the end of the list.

$$\mathit{biggers} [3, 10, 4, -2, 1, 3] = [10, 4, 3]$$

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Exercise: Is *biggers* a foldr?

Definition (Mutumorphism)

Functions f_1, \dots, f_n are said to form a mutumorphism if each f_i ($i = 1, 2, \dots, n$) is defined in the following form:

$$\begin{aligned} f_i [] &= e_i \\ f_i (a : x) &= a \oplus_i (f_1 x, f_2 x, \dots, f_n x) \end{aligned}$$

where e_i ($i = 1, 2, \dots, n$) are given constants and \oplus_i ($i = 1, 2, \dots, n$) are given binary functions. We represent the function $f x = (f_1 x, \dots, f_n x)$ as follows.

$$f = \llbracket (e_1, \dots, e_n), (\oplus_1, \dots, \oplus_n) \rrbracket.$$

Expressive Power of Mutumorphism

- *foldr* is a special case:

$$\mathit{foldr} (\oplus) e = \llbracket (e), (\mathit{oplus}) \rrbracket$$

- It covers all primitive recursive functions on lists.

$$\begin{aligned} \mathit{prim} [] &= e \\ \mathit{prim} (a : x) &= F(a, x, \mathit{prim} x) \end{aligned}$$

This is because we can *prim* is mutually defined with the identity function on lists.

biggers as a Mutumorphism

$biggers = fst \circ [([], 0), (\oplus_1, \oplus_2)]$
where $a \oplus_1 (r, s) = \text{if } a > s \text{ then } a : r \text{ else } r$
 $a \oplus_2 (r, s) = a + s$

Lemma (Mutu-Tupling)

$$\begin{aligned} & \llbracket (e_1, e_2, \dots, e_n), (\oplus_1, \oplus_2, \dots, \oplus_n) \rrbracket \\ & = \text{foldr } (\oplus) (e_1, e_2, \dots, e_n) \\ & \quad \text{where } a \oplus r = (a \oplus_1 r, a \oplus_2 r, \dots, a \oplus_n r) \end{aligned}$$

Consider, as an example, to apply the mutu-tupling lemma to *biggers*.

$$\begin{aligned}
 & \textit{biggers} \\
 = & \quad \{ \text{mutumorphism for } \textit{biggers} \} \\
 & \textit{fst} \circ \llbracket ([], 0), (\oplus_1, \oplus_2) \rrbracket \\
 = & \quad \{ \text{mutu-tupling lemma} \} \\
 & \textit{fst} \circ \textit{foldr} (\oplus) ([], 0) \\
 & \quad \textbf{where } a \oplus (r, s) = (\textbf{if } a > s \textbf{ then } a : r \textbf{ else } r, a + s)
 \end{aligned}$$

Inlining *foldr* in the derived program gives the following readable recursive program:

```
biggers x = let (r, s) = tup x in r  
  where tup [] = ([], 0)  
        tup (a : x) = let (r, s) = tup x  
                      in (if a > s then a : r else r, a + s)
```

Lemma (Foldr-Tupling)

$$(foldr (\oplus_1) e_1 x, foldr (\oplus_2) e_2 x) = foldr (\oplus) (e_1, e_2) x$$

where $a \oplus (r_1, r_2) = (a \oplus_1 r_1, a \oplus_2 r_2)$

For example, the following program for computing the average of a list:

$$\text{average } x = \text{sum } x / \text{length } x$$

can be transformed into the following with the foldr-tupling lemma.

$$\begin{aligned} \text{average } x = & \mathbf{let} (s, l) = \text{tup } x \mathbf{in} s / l \\ & \mathbf{where} \text{ tup} = \text{foldr } (\lambda a (s, l). (a + s, 1 + l)) (0, 0) \end{aligned}$$

Homework BMF 3-2

(1) Using tupling transformation to derive an efficient program for computing *tailsums*.

$$\begin{aligned} \text{tailsums } [] &= [0] \\ \text{tailsums } (a : x) &= \text{tailsums } x ++ [a + \text{sum } x] \end{aligned}$$

(2) Code the efficient program in Haskell.