Chapter 26. Automatic Parallelization

– An Application –

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Outline

1 Review: Foldr

2 Unfold

Maximum Prefix Sum Problem

Design a D&C parallel program that computes the maximum of all the prefix sums of a list.

$$mps\ [\underline{1, -2, 3, -9, 5, 7}, -10, 8, -9, 10] = 5$$

Review: List Homomorphism

Function h on lists is a list homomorphism, if

$$\begin{array}{lcl} h\,[\,] & = & e \\ h\,[a] & = & f\,a \\ h\,(x \mathbin{+\!\!+} y) & = & h\,x \odot h\,y \end{array}$$

for some $\odot$.

Properties

- Suitable for parallel computation in the D&C style
- Basic concept for skeletal parallel programming
- Enjoy many nice algebraic properties (1st, 2nd, 3rd Homomorphism theorems)

Review: Existence of Homomorphism

Existence Lemma

The list function h is a homomorphism iff the implication

$$h\ v = h\ x \ \wedge\ h\ w = h\ y\ \Rightarrow\ h\ (v +\!\!+ w) = h\ (x +\!\!+ y)$$

holds for all lists v, w, x, y.

The Third Homomorphism Theorem (Gibbons:JFP95)

A function f can be described as a foldl and a foldr

$$
\begin{aligned}
h &= \oplus \not\leftarrow_e \\
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iff there exists an associative operator $\odot$ such that

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Two sequential programs guarantee existence of a parallel program!

Proof of the Third Homomorphism Theorem

Proof. Let $h\ v = h\ x$ and $h\ w = h\ y$. Then:

$$
\begin{aligned}
& h\ (v +\!\!+\ w) \\
= & \quad \{\ h = \oplus \not\leftarrow_e\ \} \\
& \oplus \not\leftarrow_e (v +\!\!+\ w) \\
= & \quad \{\ \text{property of right-to-left reduction}\ \} \\
& \oplus \not\leftarrow_{\oplus \not\leftarrow_e w} v \\
= & \quad \{\ h\ w = h\ y\ \} \\
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& \oplus \not\leftarrow_e (v +\!\!+\ y) \\
= & \quad \{\ h = \oplus \not\leftarrow_e\ \} \\
& h\ (v +\!\!+\ y) \\
= & \quad \{\ \text{symmetrically, since } h = \otimes \not\rightarrow_e\ \} \\
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By the Existence Lemma, h is a homomorphism.

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$$\begin{array}{lcl} psums\ (a:x) & = & a : (a+) * (psums\ x) \\ psums\ (x +\!\!+ [b]) & = & psums\ x +\!\!+ [last\ (psums\ x) + b] \end{array}$$

A Challenge Problem

It remains as a challenge to automatically derive *efficient* an associative operator $\odot$ from $\oplus$ and $\otimes$.

Parallelization Theorem

Let f° denote a weak right inverse of f.

$$\frac{\begin{array}{rcl} f(a : x) & = & a \oplus f\ x \\ f(x +\!\!+ [b]) & = & f\ x \otimes b \end{array}}{\begin{array}{l} f(x +\!\!+ y) = f\ x \odot f\ y \\ \quad \textbf{where} \quad a \odot b = f(f^\circ\ a +\!\!+ f^\circ\ b) \end{array}}$$

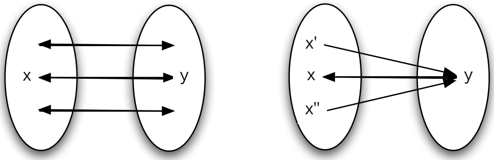
Weak (Right) Inverse

- g is an inverse of f, if

$$g\ y = x \ \Leftrightarrow\ f\ x = y$$

- g is a weak (right) inverse of f, if for $y \in \mathrm{image}(f)$

$$g\ y = x \ \Rightarrow\ f\ x = y$$



Properties of Weak Inverse

- Weak inverse always exists but may not be unique.

Example: Function *sum*

$$\begin{array}{lcl} sum\ [\,] & = & 0 \\ sum\ (a : x) & = & a + sum\ x \end{array}$$

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$$\begin{array}{lcl} g_1\ y & = & [y] \\ g_2\ y & = & [0, y] \\ & \cdots & \end{array}$$

Parallelizing *sum*

From

1. $sum\ (a : x) = a + sum\ x$
2. $sum\ (x +\!\!+ [b]) = sum\ x + b$
3. $sum^{\circ}\ y = [y]$

we soon obtain

$$sum\ (x +\!\!+ y) = sum\ x \odot sum\ y$$

where

$$\begin{aligned} a \odot b &= sum\ (sum^{\circ}\ a +\!\!+ sum^{\circ}\ b) \\ &= sum\ ([a] +\!\!+ [b]) \\ &= a + b \end{aligned}$$

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Weak inversion is not easy!

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- What is it for *mps*?

$$\begin{array}{l} mps\ [\,] = 0 \\ mps\ (a : x) = 0 \uparrow a \uparrow (a + mps\ x) \end{array}$$

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$$f\ x = (mps\ x, sum\ x)$$

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- What is it for $f = mps \vartriangle sum$? $\underline{f^\circ\ (p, s) = [p, s - p]}$

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Weak inversion is challenging

Can you find a weak inverse for f?

$$f\ x = (mss\ x, mps\ x, mts\ x, sum\ x)$$

where

$$\begin{array}{lcl}
mss\ [\,] & = & 0 \\
mss\ (a : x) & = & (a + mps\ x) \uparrow mss\ x \uparrow 0 \\
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$$\underline{f^{\circ}\ (m, p, t, s) = [p, s - p - t, m, t - m]}$$

Derivation of Weak Right Inverse

- Idea:

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- Consider to find a weak right inverse for f defined by

$$f\ x = (mps\ x, sum\ x)$$

Let x_1, x_2 be a solution to the following equations:

$$\begin{array}{rcl} mps\ [x_1, x_2] & = & p \\ sum\ [x_1, x_2] & = & s \end{array}$$

then

$$f^{\circ}\ (p, s) = [x_1, x_2]$$

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Conditional Linear Equations

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\begin{array}{rcl}
t_1(x_1, x_2, \ldots, x_m) & = & c_1 \\
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t_m(x_1, x_2, \ldots, x_m) & = & c_m
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$$
\begin{array}{lcl}
t & ::= & n \mid x \mid n\ x \mid t_1 + t_2 \mid p \to t_1; t_2 \\
p & ::= & t_1 < t_2 \mid t_1 = t_2 \mid \neg p \mid p_1 \wedge p_2 \mid p_1 \vee p_2
\end{array}
$$

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Conditional linear equations can be efficiently solved by using Mathematica. [PLDI'07]

Can we generalize the idea from lists to trees?

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$$\Downarrow$$

$$\frac{\begin{array}{l} \text{f is a bottom-up tree reduction} \\ \text{f is a top-down tree reduction} \end{array}}{\begin{array}{l} f(t_1 \triangleleft t_2) = f\,t_1 \odot f\,t_2 \\ \quad \textbf{where} \quad a \odot b = f(f^{\circ}\ a \triangleleft f^{\circ}\ b) \end{array}}$$

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Yes, see POPL'09. In fact, all the ideas in this course can be naturally generated to trees.